

Dihedral ‘star’ tensegrity structures

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ARTICLE INFO

Article history:

Received 26 September 2008

Received in revised form 10 May 2009

Available online 13 June 2009

Keywords:

Tensegrity structure
Dihedral symmetry
Block-diagonalisation
Self-equilibrium
Stability

ABSTRACT

This paper presents conditions for self-equilibrium and super stability of dihedral ‘star’ tensegrity structures, based on their dihedral symmetry. It is demonstrated that the structures are super stable if and only if they have an odd number of struts, and the struts are as close as possible to each other. Numerical investigations show that their prestress stability is sensitive to the geometry realisation.

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1. Introduction

In this paper, we describe the equilibrium and stability of dihedral ‘star’ tensegrity structures which we derive from the classic dihedral prismatic tensegrity structures. Both of these two classes of structures are of dihedral symmetry, and therefore, the stability properties of the star structures can be investigated by the methods for the prismatic structures in our early studies (Zhang et al., 2009a,b).

The horizontal cables in each of the two parallel circles containing the nodes in a prismatic structure are replaced by a star of cables in a ‘star’ structure, with a new centre node. An example ‘star’ structure is shown in Fig. 1(b), along with the parent prismatic structure in Fig. 1(a). Also shown in Fig. 1(c) is a modified version of the structure, where there exists a centre member connected to the two centre nodes.

There is a clear link between the star structures, and the parent prismatic structures that were studied by Connelly and Terrell (1995) and Zhang et al. (2009a). Indeed, we shall see that the equilibrium positions of the nodes, and self-stress forces in the vertical cables and the struts, are identical in the star and prismatic structures, as long as there is no centre member. However, the star structure has many more infinitesimal mechanisms than its parent prismatic structure: at each of the boundary nodes, a strut is in equilibrium with two cables, all of which must therefore lie in a

plane; thus, out-of-plane movement of the node must be an infinitesimal mechanism, and there are at least six infinitesimal mechanisms – in fact there is another infinitesimal mechanism corresponding to the existence of one self-stress mode of the structure. By contrast, there is only one infinitesimal mechanism in the prismatic tensegrity structure. Despite this, we will show that many dihedral star tensegrity structures can be stable, and further, that in some cases they are *super* stable, which implies that they are stable for any level of self-stress, independently of the stiffness of the members.

Following this introduction, the paper is organized as follows: Section 2 uses the symmetry of a star structure to find its configuration and self-stress forces in the state of self-equilibrium. Section 3 presents the necessary and sufficient condition for an ‘indivisible’ structure. Section 4 block-diagonalises the force density matrix and finds the condition, in terms of connectivity of vertical cables, for super stability of the star structures; prestress stability of the structures that are not super stable is numerically investigated. Section 5 briefly concludes the study on the star structures, and discusses the stability properties of those with centre members.

2. Configuration

In this section, we introduce the connectivity and geometry of a general star structure, and find the internal forces that equilibrate every node. The structure has dihedral symmetry, and this symmetry allows us to calculate symmetric state of self-stress by considering the equilibrium equations of only representative nodes.

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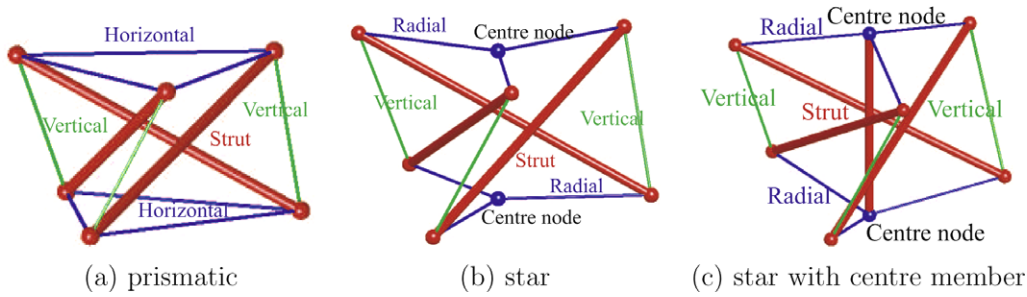


Fig. 1. Tensegrity structures that are of the same dihedral symmetry D_3 . The thick lines represent struts, and the thin lines represent cables.

2.1. Symmetry and connectivity

We are considering star tensegrity structures that have dihedral symmetry, denoted by D_n (in the Schoenflies notation, see for example Kettle (1995)): there is a single major n -fold rotation (C_n^1) axis, which we assume is the vertical, z -axis, and n twofold rotation (C_{2j}) axes perpendicular to this major axis. In total there are $2n$ symmetry operations. A star structure has the same appearance before and after the transformation by applying any of these symmetry operations.

Consider a specific set of elements (nodes or members) of a structure with symmetry G . If one element in a set can be transformed to all of the other elements of that set by the symmetry operations in G , then this set of elements are said to belong to the same orbit. A structure can have several different orbits of elements.

In contrast to prismatic structures, which have only one orbit of nodes, there are two orbits of nodes in star structures – boundary nodes and centre nodes, as shown in Fig. 2:

- There are $2n$ ‘boundary’ nodes arranged in two horizontal circles of radius R around the vertical z -axis; there is a one-to-one correspondence between the boundary nodes and the symmetry operations. (When there is a one-to-one correspondence between elements and symmetry operations, the orbit is called a regular orbit.)
- There are two ‘centre’ nodes that lie on the centres of the two horizontal circles; the cyclic (n -fold) rotation operations do not change the locations of these nodes, while the twofold rotation operations swap their positions.

Thus, there are in total $2n + 2$ nodes. The two horizontal circles containing the boundary nodes are at $z = \pm H/2$, and the centre nodes are also at $z = \pm H/2$, as shown in Fig. 2(c).

There are three orbits of members: radial cables, vertical cables and struts. The members in each orbit have the same length and internal force, and therefore, the same force density (ratio of internal force to length). Each of the boundary nodes in a circle is con-

nected by a ‘radial’ cable to a centre node. Hence, there are $2n$ radial cables, and each symmetry operation transforms a radial cable into one of the other radial cables; i.e., there is a one-to-one correspondence between the radial cables and the symmetry operations (the radial cables form a regular orbit). Each boundary node is connected by a strut and a ‘vertical’ cable to boundary nodes in the other circle. Thus, there are only n vertical cables, and n struts: there is a one-to-two correspondence between the vertical cables (or struts) and the symmetry operations. Each vertical cable and strut intersects one of the twofold horizontal rotation axes, and this twofold operation transforms this vertical cable (or strut) into itself.

It is possible to have different connectivities of the vertical cables and struts for any $n > 3$. We use the notation D_n^v to describe the connectivity of a star tensegrity with D_n symmetry, where v describes the connectivity of the vertical cables, assuming that connectivity of struts is fixed. The boundary nodes in the upper and lower circles are, respectively, numbered as N_0, N_1, \dots, N_{n-1} and $N_n, N_{n+1}, \dots, N_{2n-1}$, and the upper and lower centre nodes are numbered as N_{2n} and N_{2n+1} , respectively. We describe the connectivity of a reference node N_0 as follows – all other connections are then defined by the symmetry.

- (1) Without loss of generality, we assume that a strut connects node N_0 in the upper circle to node N_n in the lower circle.
- (2) A radial cable in the upper circle connects node N_0 to the centre node N_{2n} , and a radial cable in the lower circle connects node N_n to the centre node N_{2n+1} .
- (3) A vertical cable connects node N_0 in the upper circle to node N_{n+v} in the lower circle. We restrict $1 < v < n/2$ (choosing $n/2 < v < n$ would give essentially the same set of structures, but in a reflection symmetry with respect to the plane $z = 0$).

The numbering of nodes of two example structures with D_5 symmetry, D_5^1 and D_5^2 , is shown in Fig. 3. Node N_0 is connected by a strut to node N_5 , and by a vertical cable to node N_6 for D_5^1 , and to node N_7 for D_5^2 .

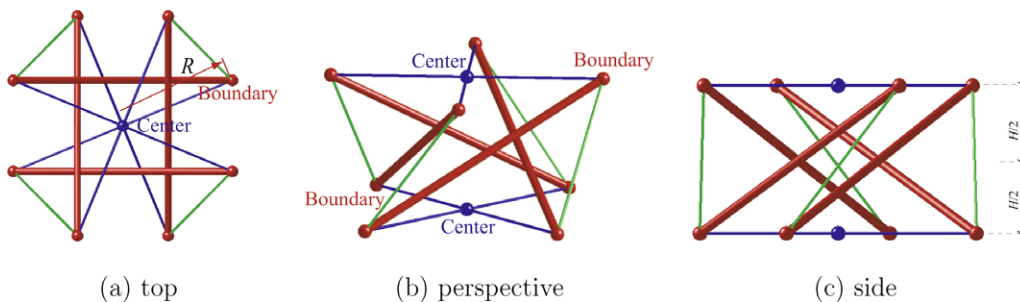


Fig. 2. The dihedral star tensegrity structure D_4^1 . R and H are the radius of the circle of boundary nodes and height of the structure, respectively.

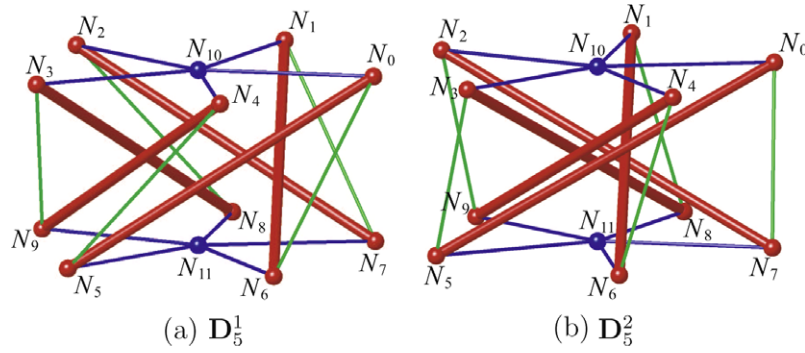


Fig. 3. The nodal numbering of two example structures with D_5 symmetry.

2.2. Symmetric state of self-stress

Because of the high symmetry of the star structures, we only need to consider the equilibrium of one reference node from each orbit, to find the symmetric state of self-stress. Thus we consider the equilibrium of one boundary node, and one centre node, in the absence of external forces.

Consider the boundary nodes first. Take one of them, for example node N_0 in the upper circle, as the *reference node* and let $\mathbf{x}_0 \in \mathfrak{R}^3$ denote its coordinates in three-dimensional space. The coordinates of the other two boundary nodes in the lower circle, connected to the reference node by the strut and vertical cable, respectively, are denoted by \mathbf{x}_s and \mathbf{x}_v ; the coordinates of the centre node in the upper circle is denoted by \mathbf{x}_c .

Since the boundary nodes are in the same orbit, the reference node \mathbf{x}_0 can be transformed to the boundary nodes \mathbf{x}_s and \mathbf{x}_v by the proper twofold rotations written in the form of transformation matrices \mathbf{R}_s and \mathbf{R}_v :

$$\begin{aligned} \mathbf{x}_s &= \mathbf{R}_s \mathbf{x}_0, \\ \mathbf{x}_v &= \mathbf{R}_v \mathbf{x}_0, \end{aligned} \tag{1}$$

where \mathbf{R}_s and \mathbf{R}_v are defined as

$$\mathbf{R}_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{R}_v = \begin{bmatrix} C_v & S_v & 0 \\ S_v & -C_v & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{2}$$

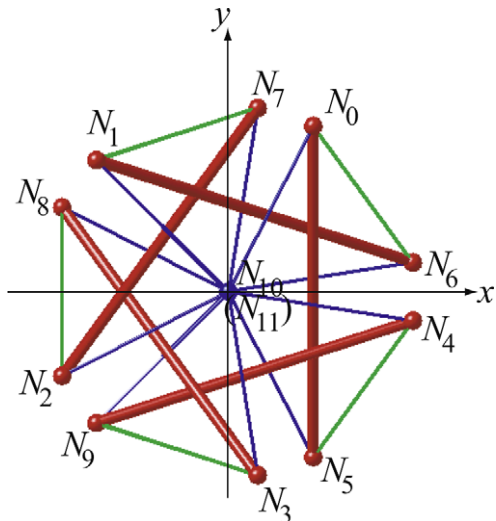


Fig. 4. Top view of the structure D_5^1 . The strut connected to reference node N_0 and node N_5 intersects the x -axis.

using the notation $C_v = \cos(2v\pi/n)$ and $S_v = \sin(2v\pi/n)$. Note that here we have effectively chosen a counter-clockwise rotation about z -axis as the positive direction for rotation of boundary nodes by our choice of \mathbf{R}_v , and we have chosen that the reference node must be connected to a strut that intersects the x -axis by our choice of \mathbf{R}_s . See an example structure D_5^1 in Fig. 4.

The coordinates \mathbf{x}_c of the centre node in the upper circle is already known, if the height H of the structure is given:

$$\mathbf{x}_c = \begin{bmatrix} 0 \\ 0 \\ H/2 \end{bmatrix}. \tag{3}$$

Denote the force densities (internal force to length ratios) of the strut, vertical and radial cables as q_s, q_v and q_r , respectively. The equilibrium of the reference node, in the absence of external force, is

$$q_s(\mathbf{x}_s - \mathbf{x}_0) + q_v(\mathbf{x}_v - \mathbf{x}_0) + q_r(\mathbf{x}_c - \mathbf{x}_0) = \mathbf{0}. \tag{4}$$

From Eqs. (1) and (2), Eq. (4) can be rewritten as

$$\bar{\mathbf{E}}\mathbf{x}_0 + q_r\mathbf{x}_c = \mathbf{0}, \tag{5}$$

where **

$$\begin{aligned} \bar{\mathbf{E}} &= \begin{bmatrix} \bar{\mathbf{E}}_1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{E}}_2 \end{bmatrix} \\ &= - \begin{bmatrix} q_v(1-C_v) + q_r & -q_v S_v & 0 \\ -q_v S_v & 2q_s + q_v(C_v + 1) + q_r & 0 \\ 0 & 0 & 2q_s + 2q_v + q_r \end{bmatrix}, \end{aligned} \tag{6}$$

where $\bar{\mathbf{E}}_1 \in \mathfrak{R}^{2 \times 2}$ and $\bar{\mathbf{E}}_2 \in \mathfrak{R}^{1 \times 1}$. Note that $\bar{\mathbf{E}}$ is in fact a part of the symmetry-adapted force density matrix presented later in the paper.

Using Eqs. (3) and (6), Eq. (5) can be separated into the following two independent equations

$$\bar{\mathbf{E}}_1 \bar{\mathbf{x}}_0 = \mathbf{0}, \tag{7}$$

and

$$\bar{\mathbf{E}}_2 H/2 + q_r H/2 = (-2q_s - 2q_v - q_r)H/2 + q_r H/2 = 0, \tag{8}$$

where the vector $\bar{\mathbf{x}}_0 \in \mathfrak{R}^2$ denotes the coordinates of the reference node in xy -plane. Because $H \neq 0$, Eq. (8) gives

$$q_v = -q_s. \tag{9}$$

In order to have non-trivial coordinates ($\mathbf{x}_0 \neq \mathbf{0}$) in xy -plane, $\bar{\mathbf{E}}_1$ should be singular; i.e., its determinant should be zero. Hence, we have

$$q_r^2 + 2q_v^2(C_v - 1) = 0, \tag{10}$$

where Eq. (9) has been applied. The force densities of the cables should be positive, hence we have

$$q_r = +q_v \sqrt{2(1 - C_v)}. \quad (11)$$

Thus we have found force densities in the members that allow the structure to be in self-equilibrium – equilibrium of the centre nodes is automatically satisfied. For the reference node, the coordinate \mathbf{x}_0 in the xy -plane lies in the nullspace of $\bar{\mathbf{E}}_1$ and the coordinate in z -direction is $H/2$, giving

$$\mathbf{x}_0 = \frac{R}{R_0} \mathbf{r} + \frac{H}{2} \mathbf{h} = \frac{R}{R_0} \begin{bmatrix} C_v - 1 + \sqrt{2 - 2C_v} \\ S_v \\ 0 \end{bmatrix} + \frac{H}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (12)$$

where $R_0 (= 2\sqrt{(1 - C_v)(1 - S_{v/2})})$ is the norm of \mathbf{r} . Thus, R is the arbitrary radius of the circles containing the boundary nodes. Note that the equilibrium configuration, in terms of nodal coordinates, is identical to that of the prismatic structure with the same dihedral symmetry, and therefore, the ‘twist angel’ $\theta (= \pi/2 + v\pi/n)$ of the struts derived by Connolly and Terrell (1995) is also applicable to the case of star structures.

3. Divisibility

The previous section has found equilibrium configurations for dihedral star tensegrity structures, but these structures may be stable or unstable. Before stability investigation in the next section, we present the condition for indivisible structures, since the divisible structures should have been considered in the simpler substructures with less nodes and members.

Note that, unlike prismatic tensegrity structures catalogued by Zhang et al. (2009a), the ‘star’ tensegrity structures are never strictly divisible – all elements of the structure are connected. Despite this, it is possible for parts of the structure to act independently of one another of one particular set of relative motions – rotations around the z -axis. Thus we define a dihedral star tensegrity to be *divisible* if the members and nodes can be separated into two or more identical substructures that are only interconnected by being pinned together at the common centre nodes. Rotation of one substructure about z -axis has no mechanical influence on other substructures. Hence, the divisible structure has a finite mechanism, and therefore cannot be stable. As an example, the structure \mathbf{D}_8^2 shown in Fig. 5(a) is divisible – it can be separated into two identical structures \mathbf{D}_4^1 as shown in Fig. 5(b) and (c). The struts and vertical cables in each substructure connect one to another to form a closed circuit, so that the substructures are indivisible.

From the above discussions, we know that divisibility of a star structure is determined only by the connectivity manner of the vertical cables. Moreover, divisibility condition for vertical cables for star structures is actually identical to that for prismatic structures as discussed by Zhang et al. (2009a). For completeness of our study on stability of star structures, this condition is repeated as follows.

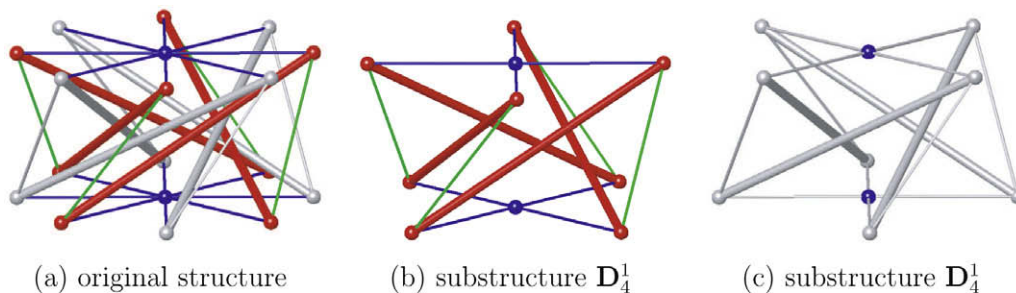


Fig. 5. Divisible dihedral star tensegrity structure \mathbf{D}_8^2 . Any of the two substructures \mathbf{D}_4^1 can rotate about the z -axis without mechanical influence on the other.

From the labels of nodes and definition of connectivities of struts and vertical cables, node N_i in the upper circle is connected to node N_{n+i+v} by a vertical cable; node N_{n+i+v} is connected to node N_{i+v} by a strut; node N_{i+v} is connected to N_{n+i+2v} by a vertical cable, and so on. Eventually, we must return back to the starting node N_i . If we stop when the linkage returns back to the starting node N_i for the first time, the boundary nodes in the upper circle in the linkage can be listed as follows

$$N_i \rightarrow N_{i+v} \rightarrow N_{i+2v} \rightarrow N_{i+jv-mn} (\equiv N_i). \quad (13)$$

The numbers j and m indicate the number of boundary nodes in the upper circle that have been visited, and the number of circuits around the z -axis, respectively.

To return to the starting node N_i , we have $i + jv - mn = i$, and hence

$$jv = mn. \quad (14)$$

We must have integer solution for j and m , which we can write as $j = n/D$, $m = v/D$, where D is any common divisor of n and v . If the structure is indivisible, we should have visited all n boundary nodes in the upper circle. Thus, the minimum solution for j for an indivisible structure is $j = n$, from which we must have that D can only be 1. In summary, the necessary and sufficient indivisibility condition for a star structure is that v and n have no common divisor except 1.

4. Stability

This section will investigate the stability of the star tensegrity structures. We will introduce the concepts and conditions of super stability and prestress stability, and show which structures are super stable, and which are prestress stable for varying ratios of R/H .

4.1. Super stability conditions for general tensegrity structures

A super stable tensegrity structure is guaranteed to be stable for any geometry realisation (i.e., any arbitrary R and H in the case of star structures), and any level of prestress, regardless of material properties. The force density matrix (sometimes called the ‘small’ stress matrix, for example in Guest (2006)) is key to super stability property of a tensegrity structure.

The force density matrix $\mathbf{E} (\in \mathfrak{R}^{(2n+2) \times (2n+2)})$ for a star structure with \mathbf{D}_n symmetry) is a symmetric matrix, defined using the force densities: let \mathcal{S} denote the set of members connected to free node i , the (i, j) -component $\mathbf{E}_{(ij)}$ of \mathbf{E} is given as (Zhang and Ohsaki, 2006)

$$\mathbf{E}_{(ij)} = \begin{cases} \sum_{k \in \mathcal{S}} q_k & \text{for } i = j, \\ -q_k & \text{if nodes } i \text{ and } j \text{ are connected by member } k, \\ 0 & \text{if nodes } i \text{ and } j \text{ are not connected,} \end{cases} \quad (15)$$

deficient by one. To ensure a super stable structure, these blocks must be positive semi-definite and the other blocks must be positive definite. This subsection investigates when this is the case.

$\tilde{\mathbf{E}}^{A_1}$ and $\tilde{\mathbf{E}}^{A_2}$ have eigenvalues

$$\lambda_1^{A_1} = \lambda_1^{A_2} = 0 \quad \text{and} \quad \lambda_2^{A_1} = \lambda_2^{A_2} = (n+1)q_r > 0, \quad (24)$$

and hence they are positive semi-definite.

B_1 and B_2 exist only for n even, and v is odd for an indivisible structure. Hence, substituting Eqs. (9) and (11) to Eq. (23), we have

$$\begin{aligned} \lambda^{B_1} &= \tilde{\mathbf{E}}^{B_1} = q_r - q_s + (-1)^{v+1}q_v = q_r - q_s + q_v \\ &= q_v(2 + \sqrt{2(1-C_v)}), \end{aligned} \quad (25)$$

and

$$\begin{aligned} \lambda^{B_2} &= \tilde{\mathbf{E}}^{B_2} = q_r + q_s + (-1)^v q_v = q_r + q_s - q_v \\ &= q_v(-2 + \sqrt{2(1-C_v)}). \end{aligned} \quad (26)$$

Thus $\tilde{\mathbf{E}}^{B_1} > 0$, and is positive definite, while $\tilde{\mathbf{E}}^{B_2} \leq 0$. In fact the equality only holds if $v = n/2$, and hence for an indivisible structure ($v \neq n/2$), $\tilde{\mathbf{E}}^{B_2} < 0$, and is thus negative definite. Therefore, for n even, dihedral star tensegrity structures are never super stable. We will now consider indivisible structures with n odd to find the super stability condition. The two eigenvalues of $\tilde{\mathbf{E}}^{E_k}$ are

$$\begin{aligned} \frac{1}{q_v} \lambda_1^{E_k} &= \sqrt{2(1-C_v)} + \sqrt{2(1-C_{kv})} > 0, \\ \frac{1}{q_v} \lambda_2^{E_k} &= \sqrt{2(1-C_v)} - \sqrt{2(1-C_{kv})}. \end{aligned} \quad (27)$$

For $k = 1$, we have $\lambda_2^{E_1} = 0$ as expected for the equilibrium condition. Thus, for $n = 3$ where $k > 1$ does not exist, the dihedral star tensegrity structure is super stable.

For $n > 4$, we must consider $\tilde{\mathbf{E}}^{E_k}$ for $k > 1$. For a super stable tensegrity, $\tilde{\mathbf{E}}^{E_k}$ for all $1 < k \leq p = (n-1)/2$ must be positive definite; i.e., $\lambda_2^{E_k}$ must be positive; and hence, from Eq. (27) we require $C_{kv} > C_v$, for all $1 < k \leq (n-1)/2$. (28)

Each of C_{jv} ($j \in \{1, \dots, (n-1)/2\}$) takes one of the value in the following list with n elements

$$\left\{ \cos \frac{1}{n} 2\pi, \cos \frac{2}{n} 2\pi, \dots, \cos \frac{n}{n} 2\pi \right\}. \quad (29)$$

The nine cosine values, four of which duplicate, for the case $n = 9$ is illustrated in Fig. 6, where the horizontal and vertical axes, respectively, denote cosine and sine values of a specific angle θ .

Note that $\cos \frac{1}{n} 2\pi = \cos \frac{n-1}{n} 2\pi$; moreover, $C_{jv} \neq \cos 2\pi$ holds, because v and n have no common divisor except 1 for an indivisible structure and $1 < j \leq (n-1)/2$. Thus, the

$$\left\{ \cos \frac{1}{n} 2\pi, \cos \frac{2}{n} 2\pi, \dots, \cos \frac{n-1}{2n} 2\pi \right\}. \quad (30)$$

where

$$\left\{ \cos \frac{1}{n} 2\pi > \cos \frac{2}{n} 2\pi > \dots > \cos \frac{n-1}{2n} 2\pi \right\}. \quad (31)$$

It is apparent that $v = (n-1)/2 = p$ will lead to $C_{kv} \geq C_v$ for $1 < k \leq p$. However, this is the necessary and sufficient condition only if each of C_{jv} ($j \in \{1, \dots, (n-1)/2\}$) has one-to-one correspondence to the elements in the condensed list in Eq. (30), as proved in Lemma 1.

Lemma 1. Each of C_{jv} ($j \in \{1, \dots, (n-1)/2\}$) has one-to-one correspondence with the elements in the condensed list in Eq. (30), for an indivisible structure with n odd.

Proof. To prove the lemma, we need only to show that the relation $C_{k_1 v} \neq C_{k_2 v}$ holds for $k_1 \neq k_2$.

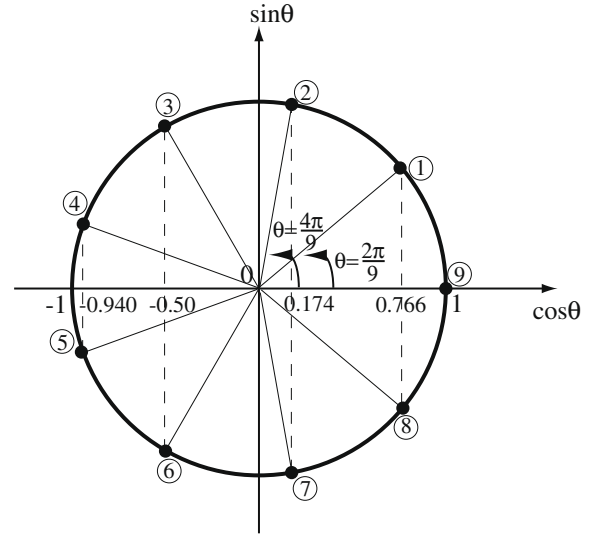


Fig. 6. Cosine corresponding to the connectivity of vertical cables $v(n=9)$. list can be condensed as follows with $p = (n-1)/2$ elements

The relation $C_{k_1 v} = C_{k_2 v}$ holds only if

$$\frac{k_1 v - m_1 n}{n} 2\pi + \frac{k_2 v - m_2 n}{n} 2\pi = 2\pi, \quad (32)$$

where m_1 and m_2 are the integers satisfying $1 \leq k_1 v - m_1 n \leq n$ and $1 \leq k_2 v - m_2 n < n$. Thus we have

$$(k_1 + k_2)v = (m_1 + m_2 + 1)n. \quad (33)$$

Since v and n have no common divisor except 1 for an indivisible structure, the smallest possible (integer) solution for k_1 and k_2 is $k_1 + k_2 = n$. However, we have

$$2 \leq k_1 + k_2 \leq n-1 < n, \quad (34)$$

due to $1 \leq k_1 \leq (n-1)/2$ and $1 \leq k_2 \leq (n-1)/2$. Therefore, Eq. (33) can not hold, and hence $C_{k_1 v} \neq C_{k_2 v}$ holds for $k_1 \neq k_2$.

As there are in total $(n-1)/2$ elements for $j \in \{1, \dots, (n-1)/2\}$, and $(n-1)/2$ different values in Eq. (30), every cosine value in the list has been taken, but only once, by C_{jv} for an indivisible structure with n odd, which proves the lemma. \square

Some examples are given in Fig. 7 for different v for the case $n = 9$. The solid lines links the points showing cosine values for C_{jv} ($j \in \{1, \dots, 4\}$). The four points take different cosine values for indivisible structures ($v = 1, 2, 4$), and duplicate for the divisible structure ($v = 3$).

From Lemma 1, we must have the following relation from Eq. (30) to ensure that the relation in Eq. (28) always holds

$$v = p = \frac{n-1}{2}. \quad (35)$$

In other words, a dihedral star tensegrity structure is super stable if and only if it has odd number of struts (n odd), and the struts are as close to each other as possible ($v = (n-1)/2$). This is the necessary and sufficient condition for super stability of dihedral star tensegrity structures. By contrast, super stability of a prismatic structure is not related to connectivity manner of its vertical cables: it is super stable if and only if its horizontal cables are connected to adjacent nodes (Connelly and Terrell, 1995).

4.4. Prestress stability of dihedral star structures

We have used analytical method to find the super stability condition for dihedral star tensegrity structures in the previous sec-

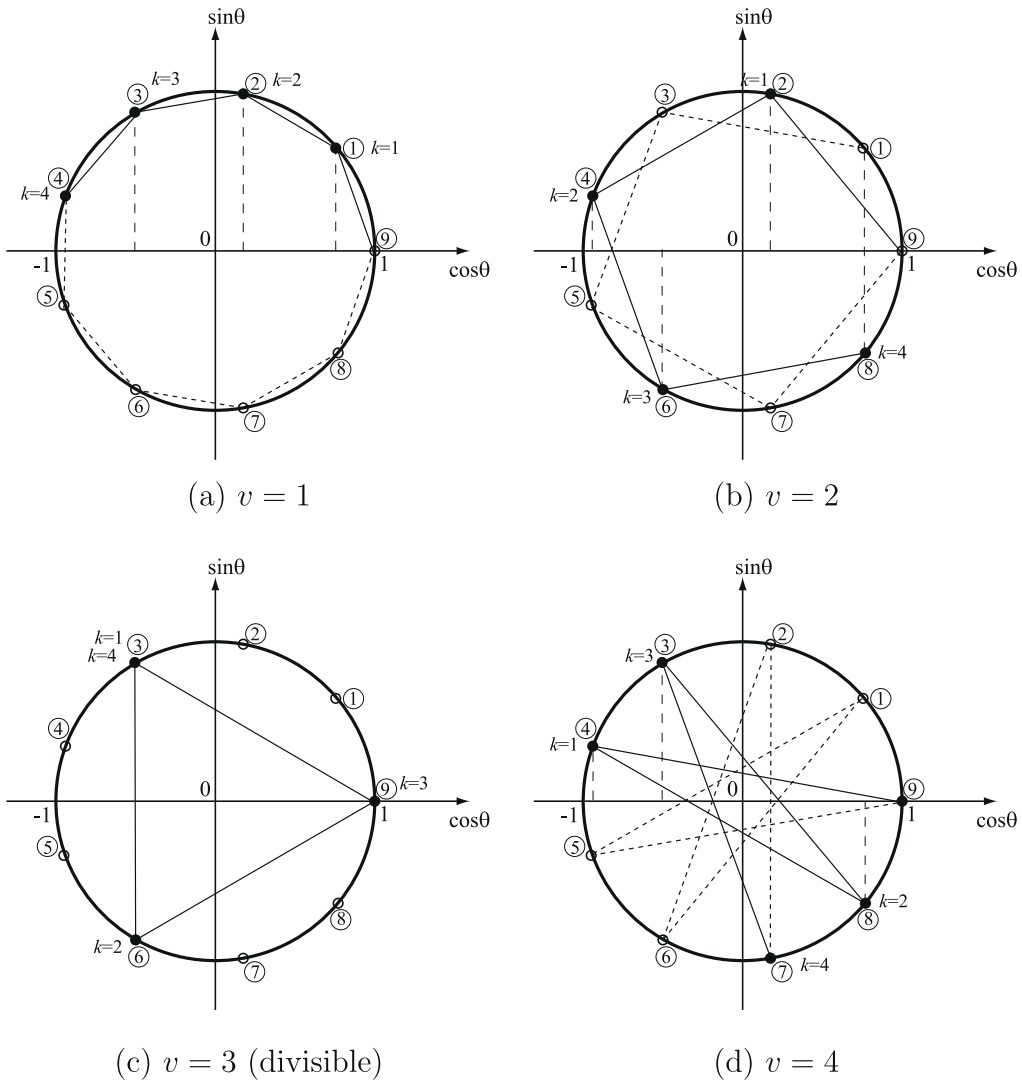


Fig. 7. Connectivity of boundary nodes in one circle through struts and vertical cables ($n = 9$). Every cosine value is taken only once, if the structure is indivisible, see the cases (a), (b) and (d). For the divisible case in (c), only part of the cosine values have been taken.

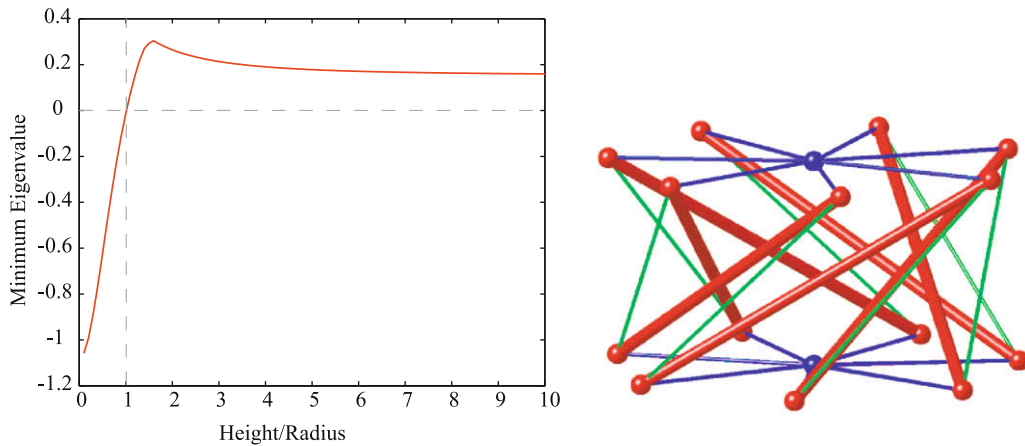


Fig. 8. Prestress stability of the dihedral star tensegrity structure D_7^1 . When the height/radius ratio is larger than 1.02, it is prestress stable with the positive minimum eigenvalue of the reduced stiffness matrix.

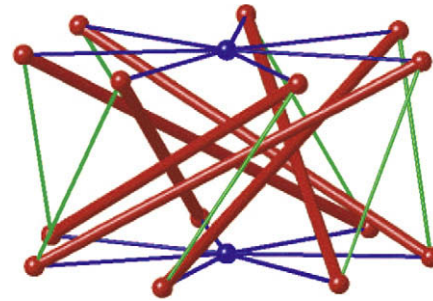
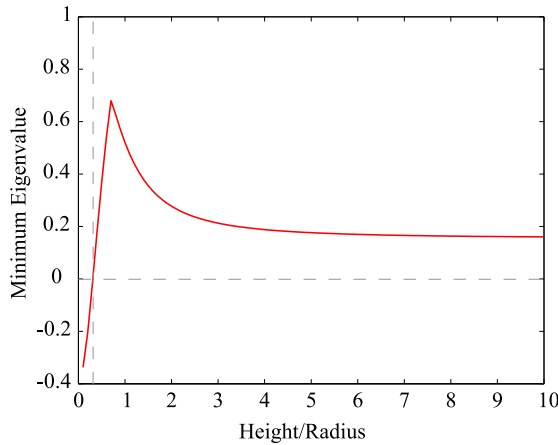


Fig. 9. Prestress stability of the dihedral star tensegrity structure \mathbf{D}_7^2 . When the height/radius ratio is larger than 0.32, it is prestress stable with the positive minimum eigenvalue of the reduced stiffness matrix.

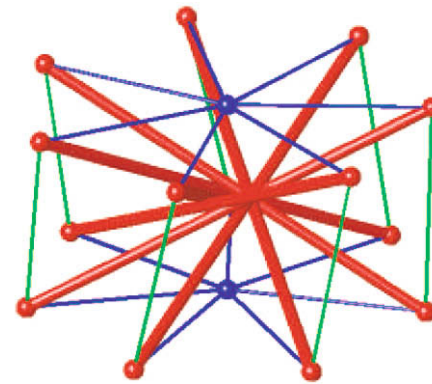
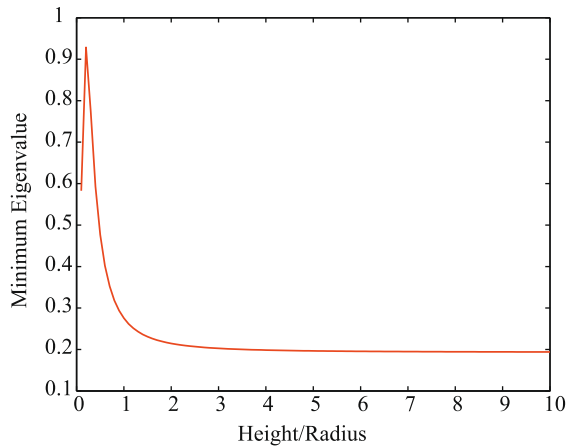


Fig. 10. Prestress stability of the dihedral star tensegrity structure \mathbf{D}_7^3 . From the super stability condition for star structures, this structure is super stable, and hence, it is always prestress stable irrespective of the height/radius ratio.

tion. In this section, we will show through numerical calculations that some other structures, that are not super stable, can still be prestress stable if certain conditions are satisfied – a treatment that follows Zhang et al. (2009a).

A prestress stable structure has a positive definite reduced geometrical stiffness matrix

$$\mathbf{Q} = \mathbf{M}\mathbf{K}^G\mathbf{M}^T, \tag{36}$$

where columns of \mathbf{M} are the infinitesimal mechanisms of the structure, and \mathbf{K}^G is the geometrical stiffness matrix defined by the tensor product of a 3×3 identity matrix \mathbf{I}_3 and the force density matrix, $\mathbf{K}^G = \mathbf{I}_3 \otimes \mathbf{E}$ (Guest, 2006). The structure is prestress stable if and only if the minimum eigenvalue λ^Q of \mathbf{Q} is positive.

As an example, Figs. 8–10 plot the values of λ^Q against the ratios of height to radius (H/R) for the star tensegrity structures with dihedral symmetry \mathbf{D}_7 . The force density matrix is calculated relative to the force density of vertical cables, or by assigning $q_v = 1$ alternatively without losing generality. The structure \mathbf{D}_7^3 is super stable and is thus always prestress stable; the structures \mathbf{D}_7^1 and \mathbf{D}_7^2 are not super stable, but it can be observed from Figs. 8 and 9 that they are prestress stable if the height/radius ratio is large enough.

These figures have the same appearance: the minimum eigenvalue of the reduced stiffness matrix increases sharply firstly, and then decreases with the increasing height/radius ratios. Final-

ly, the minimum eigenvalue converges gradually to a positive value, such that the structure is prestress stable.

Table 1 shows the stability of star structures with $3 < n \leq 10$. It can be seen that every indivisible structure in this range can be prestress stable.

5. Discussion

This paper has presented the necessary and sufficient condition for super stability of dihedral star tensegrity structures – the structures are super stable if and only if they have odd number of struts, and the struts are as close to each other as possible. Furthermore,

Table 1

Stability of dihedral star tensegrity structures \mathbf{D}_n^ν , for $3 \leq n \leq 10$. ‘S’ denotes super stable; if the structure is prestress stable only when the height/radius ratio is larger than r , then this is indicated as ‘> r ’; and if the structure is divisible, its substructures are given.

ν	n							
	3	4	5	6	7	8	9	10
1	S	≤ 0.46	≤ 0.65	≤ 0.87	≤ 1.02	≤ 1.18	≤ 1.30	≤ 1.43
2		$2\mathbf{D}_2^1$	S	$2\mathbf{D}_3^1$	≤ 0.32	$2\mathbf{D}_4^1$	≤ 0.57	$2\mathbf{D}_5^1$
3				$3\mathbf{D}_2^1$	S	≤ 0.11	$3\mathbf{D}_3^1$	≤ 0.29
4						$4\mathbf{D}_2^1$	S	$2\mathbf{D}_5^2$
5								$5\mathbf{D}_2^1$

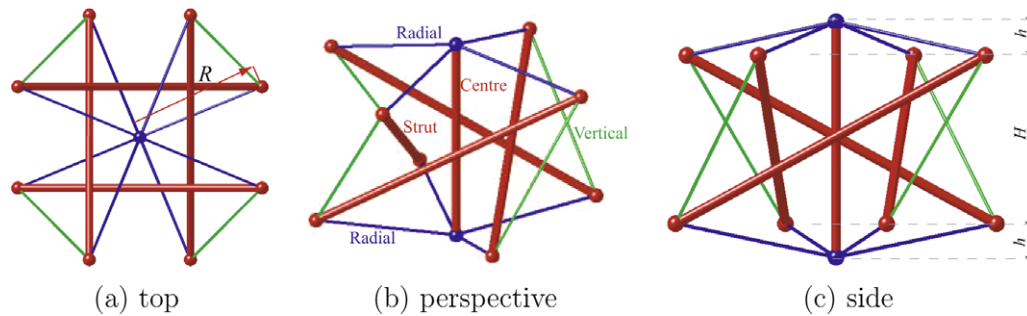


Fig. 11. Dihedral star tensegrity structure with centre member. It has the same connectivity as the structure D_4^1 , except for the additional centre member connecting the two centre nodes. The parameter h denotes the distance between a centre node to the closest centre of circle containing boundary nodes. The centre member is a strut when $h > 0$, a cable when $h < 0$, and there exists no prestress in the member when $h = 0$. This structure is prestress stable but not super stable as confirmed by numerical calculation.

we conjecture that all indivisible dihedral star tensegrity are prestress stable if the height/radius ratio is large enough: numerical calculations have shown this to be true for all $3 \leq n \leq 1000$, although only simpler cases $3 \leq n \leq 10$ have been presented.

If the centre nodes of a star structure are connected by an additional 'centre' member, see for example the structure shown in Fig. 11, the new structure is also of dihedral symmetry. Numerical calculations show that the super stability condition and the conjectures on prestress stability for the structures without centre members also apply to these structures. However, any proof of their stability properties is complicated by the existence of an additional parameter, the distance between the centre nodes.

Acknowledgements

J.Y. Zhang is grateful for financial supports from The Kyoto University Foundation and the EPSRC while visiting the Department of Engineering at the University of Cambridge, and the support from Japan Society for the Promotion of Science for his post-doctoral research at Kyoto University. R. Connelly is grateful for support from the EPSRC while visiting the Department of Engineering at the Uni-

versity of Cambridge, and support from NSF Grant No. DMS-0209595.

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